Fourth-Moment Standardization for Structural Reliability Assessment

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Abstract: In structural reliability analysis, the uncertainties related to resistance and load are generally expressed as random variables that have known cumulative distribution functions. However, in practical applications, the cumulative distribution functions of some random variables may be unknown, and the probabilistic characteristics of these variables may be expressed using only statistical moments. In the present paper, in order to conduct structural reliability analysis without the exclusion of random variables having unknown distributions, the third-order polynomial normal transformation technique using the first four central moments is investigated, and an explicit fourth-moment standardization function is proposed. Using the proposed method, the normal transformation for independent random variables with unknown cumulative distribution functions can be realized without using the Rosenblatt transformation or Nataf transformation. Through the numerical examples presented, the proposed method is found to be sufficiently accurate in its inclusion of the independent random variables which have unknown cumulative distribution functions, in structural reliability analyses with minimal additional computational effort.

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Introduction

The search for an effective structural reliability method has led to the development of various reliability approximation techniques, such as the first-order reliability method (FORM) (Hasofer and Lind 1974; Rackwitz 1976; Shinozuka 1983), the second-order reliability method (SORM) (Der Kiureghian et al. 1987; Der Kiureghian and De Stefano 1991; Cai and Elishakoff 1994), importance sampling Monte Carlo simulation (Melchers 1990; Fu 1994), first-order third-moment reliability method (FOTM) (Tichy 1994), response surface approach (Rajashekhar and Ellingwood 1993; Liu and Moses 1994), directional simulation methods (Nie and Ellingwood 2000), and others. In almost all of these methods, the basic random variables are assumed to have a known cumulative distribution (CDF) or probability density function (PDF). Based on CDF/PDF, the normal transformation (*x-u* transformation) and its inverse transformation (*u-x* transformation) are realized by using Rosenblatt transformation (Hohenbichler and Rackwitz 1981) or Nataf transformation (Liu and Der Kiureghian 1986). In reality, however, due to the lack of statistical data, the CDFs/PDFs of some basic random variables are often unknown, and the probabilistic characteristics of these variables are often expressed using only statistical moments. In such circumstances, the Rosenblatt transformation or Nataf transformation cannot be applied, and a strict evaluation of the probability of failure is not possible. Thus, an alternative measure of reliability is required.

In the present paper, the third-order polynomial normal transformation technique using the first four central moments is investigated. An explicit fourth-moment standardization function is proposed. Using the proposed method, the normal transformation for independent random variables with unknown CDFs/PDFs can be realized without using the Rosenblatt transformation or Nataf transformation. Through the numerical examples presented, the proposed method is found to be sufficiently accurate to include the independent random variables with unknown CDFs/PDFs in structural reliability analyses with minimal additional computational effort.

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Review of Reliability Method Including Random Variables with Unknown CDF/PDFs

A comprehensive framework for the analysis of structural reliability under incomplete probability information was proposed by Der Kiureghian and Liu (1986). It was an approach based on the Bayesian idea, in which the distribution is assumed to be a weighted average of all candidate distributions, where the weights represent the subjective probabilities of respective candidates. The proposed method was found to be consistent with full distribution structural safety theories, and has been used to measure structural safety under imperfect states of knowledge. However, one needs to select the candidate distributions and weights when using this method. A method of estimating complex distributions using B-spline functions has been proposed by Zong and Lam

(1998), in which the estimation of the PDF is summarized as a nonlinear programming problem.

Another way to conduct structural reliability analysis, including random variables with unknown CDFs/PDFs, is relaying the *u*-*x* and *x*-*u* transformations directly using the first few moments of the random variable, which can be easily obtained from the statistical data. This method can be divided into two routes; one is using the distribution families, and another is using polynomial normal transformation. As for the distribution families; Burr system, John system, Pearson system (Stuart and Ord 1987; Hong 1996), and Lambda distribution (Ramberg and Schmeiser 1974; Grigoriu 1983) can be used. Since the quality of approximating the tail area of a distribution is relatively insensitive to the distribution families selected (Pearson et al. 1979) and the solution of nonlinear equation is required to determine the parameters of the Burr and John systems or Lambda distribution, the Pearson system is commonly used.

Without loss of generality, a random variable x can be standardized as follows:

$$x_s = \frac{(x-\mu)}{\sigma} \tag{1}$$

where μ and σ=mean value and standard deviation of x, respectively.

For a standardized random variable x_s, f, the PDF of x_s, satisfies the following differential equation in the Pearson system (Stuart and Ord 1987)

$$\frac{1}{f}\frac{df}{dx_s} = -\frac{ax_s + b}{c + bx_s + dx_s^2} \tag{2a}$$

where

$$a = 10\alpha_{4X} - 12\alpha_{3X}^2 - 18 \tag{2b}$$

$$b = \alpha_{3X}(\alpha_{4X} + 3) \tag{2c}$$

$$c = 4\alpha_{4X} - 3\alpha_{3X}^2 \tag{2d}$$

$$d = 2\alpha_{4X} - 3\alpha_{3X}^2 - 6 \tag{2e}$$

where α_{3X} and α_{4X}=third- and fourth-dimensionless central moment, i.e., the skewness and kurtosis of x.

In Eq. (2), the parameters are easily and explicitly determined from the first four moments, and the forms of PDFs are dependent on the values of parameters a, b, c, and d. However, there are 12 kinds of PDFs in the Pearson system, and numerical integration is generally required to determine these PDFs (Zhao and Ang 2003).

As for the method that uses polynomial transformation, the third-order polynomial normal transformation method was suggested by Fleishman (1978), in which the transformation is formulated as

$$x_s = a_1 + a_2 u + a_3 u^2 + a_4 u^3 \tag{3}$$

where x_s=standardized random variable; u=standard normal random variable, and a_1, a_2, a_3, and a_4=polynomial coefficients that can be obtained by making the first four moments of the left side of Eq. (3) equal to those of the right side.

Since the form of Eq. (3) is simple if the coefficients a_1, a_2, a_3, and a_4 are known, it has several applications pertaining to structural reliability analysis. However, the determination of the four coefficients is not easy, since the solution of nonlinear equations has to be found (Fleishman 1978). Some methods to determine the polynomial coefficients are reported by Chen and Tung (2003). These include the moment-matching method (Fleishman 1978), least-square method (Hong and Lind 1996), and L-moments method (Tung 1999). Because the first four moments (mean, standard deviation, skewness, and kurtosis) having clear physical definitions are common in engineering and can be easily obtained using the sample data, the determination of the four coefficients using the first four moments will be focused on in this paper.

As described above, since the solution of nonlinear equations has to be found, the Fleishman expression is inexplicit. Thus, the second-order Fisher-Cornish expansion (Fisher and Cornish 1960) is sometimes used, which is expressed as

$$x_s = -h_3 + (1 - 3h_4)u + h_3 u^2 + h_4 u^3 \tag{4a}$$

in which

$$h_3 = \frac{\alpha_{3X}}{6}, \quad h_4 = \frac{\alpha_{4X} - 3}{24} \tag{4b}$$

One can see that Eq. (4) is in close form and is quite easy to use; however, since the first four moments of the right side of Eq. (4*a*) are not equal to those of the left side, the transformation generates relatively large errors.

Winterstein (1988) developed an expansion expressed as

$$x_s = -\tilde{k}\tilde{h}_3 + \tilde{k}(1 - 3\tilde{h}_4)u + \tilde{k}\tilde{h}_3 u^2 + \tilde{k}\tilde{h}_4 u^3 \tag{5a}$$

where

$$\tilde{h}_3 = \frac{\alpha_{3X}}{4 + 2\sqrt{1 + 1.5(\alpha_{4X} - 3)}}, \quad \tilde{h}_4 = \frac{\sqrt{1 + 1.5(\alpha_{4X} - 3)} - 1}{18} \tag{5b}$$

$$\tilde{k} = \frac{1}{\sqrt{1 + 2\tilde{h}_3^2 + 6\tilde{h}_4^2}} \tag{5c}$$

Apparently, the Winterstein formula requires $\alpha_{4X} > 7/3$ because of Eq. (5*b*).

Explicit Fourth-Moment Standardization Function

Expression of Standardization Function

It has been found that the Winterstein formula gives much improvement to the Fisher-Cornish expansion while managing to retain its simplicity and explicitness. However, as will be described later, since the differences of the first four moments between the two sides of Eq. (5*a*) are still large, the transformation is still not convincing. For obvious reasons, a transformation for use in practical engineering should be as simple and accurate as possible.

In this paper, a simple explicit fourth-moment standardization function is proposed, as is illustrated in the following equation, which was developed from a large amount of data of third- and fourth-dimensionless central moments through trial and error

$$x_s = S_u(u) = -l_1 + k_1 u + l_1 u^2 + k_2 u^3 \tag{6a}$$

where $S_u(u)$ denotes the third polynomial of u; and the coefficients l_1, k_1, and k_2 are given as

$$l_1 = \frac{\alpha_{3X}}{6(1 + 6l_2)}, \quad l_2 = \frac{1}{36}\left(\sqrt{6\alpha_{4X} - 8\alpha_{3X}^2 - 14} - 2\right) \tag{6b}$$

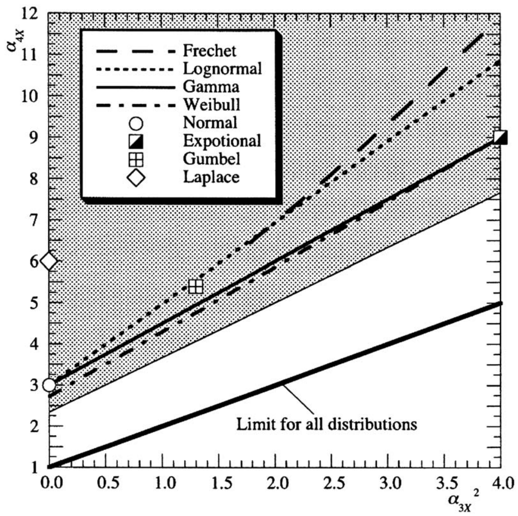


Fig. 1. Operable area of the present formula

$$k_1=\frac{1-3l_2}{(1+l_1^2-l_2^2)},\quad k_2=\frac{l_2}{(1+l_1^2+12l_2^2)} \tag{6c}$$

From Eq. (6*b*), the following condition should be satisfied:

$$\alpha_{4X}\geqslant(7+4\alpha_{3X}^2)/3 \tag{6d}$$

Using Eq. (6*d*), a lower boundary line in the $\alpha_{3X}^2-\alpha_{4X}$ plane can be depicted as shown in Fig. 1, in which the operable area of the present formula is indicated as the shaded region. In Fig. 1, the limit for all distributions expressed as $\alpha_{4X}=1+\alpha_{3X}^2$ (Johnson and Kotz 1970) is also depicted along with the $\alpha_{3X}^2-\alpha_{4X}$ relationship for some commonly used distributions, i.e., the normal, Gumbel, Laplace, and the exponential distribution, which are represented by a single point, the lognormal, the Gamma, the Frechet, and the Weibull distributions—represented by a line—are also depicted. One can see that the operable area of the present formula covers a large area in the $\alpha_{3X}^2-\alpha_{4X}$ plane, and the $\alpha_{3X}^2-\alpha_{4X}$ relationship for most commonly used distributions is in the operable area of the present formula. This implies that Eq. (6*d*) is generally operable for common engineering use.

When $\alpha_{4X}=3+4\alpha_{3X}^2/3$, one has $l_1=\alpha_{3X}/6$, $k_1=1/(1+\alpha_{3X}^2/36)$, and $l_2=k_2=0$, then Eq. (6*a*) can be expressed as

$$x_s=k_1u-\frac{1}{6}\alpha_{3X}(u^2-1)$$

when α_{3X} is small enough, $k_1\approx 1$, the equation above is simplified as

$$x_s=u-\frac{1}{6}\alpha_{3X}(u^2-1) \tag{7}$$

which becomes the first-order Fisher-Cornish expansion.

Particularly, if $\alpha_{3X}=0$ and $\alpha_{4X}=3$, then l_1, l_2, k_1, and k_2 will be obtained as $l_1=l_2=k_2=0$ and $k_1=1$, and the *u*-*x* transformation function will be degenerate as $x_s=u$.

From Eq. (6), the *x*-*u* transformation is readily obtained as

$$u=-\frac{\sqrt[3]{2}p}{\sqrt[3]{-q+\Delta}}+\frac{\sqrt[3]{-q+\Delta}}{\sqrt[3]{2}}-\frac{l_1}{3k_2} \tag{8a}$$

where

$$\Delta=\sqrt{q^2+4p^3},\quad p=\frac{3k_1k_2-l_1^2}{9k_2^2},$$
$$q=\frac{2l_1^3-9k_1k_2l_1+27k_2^2(-l_1-x_s)}{27k_2^3} \tag{8b}$$

Comparisons of Polynomial Coefficients

The four polynomial coefficients—a_1, a_2, a_3, and a_4—determined by the proposed formula are illustrated in Fig. 2, compared with those obtained using Fisher-Cornish expansion, Winterstein formula, and the accurate coefficients obtained from moment-matching method (Fleishman 1978). The coefficients are expressed as functions of α_{4X} for α_{3X}=0.0, 0.4, 0.8, and 1.2. One can clearly see from Fig. 2 that:

1. The coefficients of Fisher-Cornish expansion have the greatest differences from the accurate coefficients except when the random variable *x* is nearly a normal random variable.
2. The Winterstein formula markedly improves the Fisher-Cornish expansion and provides good results when α_{3X} is small and α_{4X} is within a particular range. However, as α_{3X} becomes larger, especially when α_{3X} is larger than 0.4, the coefficients obtained by the Winterstein formula will have significant differences compared to the accurate ones.
3. The coefficients obtained using the proposed formula are in close agreement with the accurate ones throughout the entire investigation range.

As described above, the parameters of an accurate fourth-moment standardization function should make the first four moments of the function $S_u(u)$ [the right side of Eq. (6*a*)] be equal to those of the original random variable [the left side of Eq. (6*a*)]. For the given pair value of α_{3X} and α_{4X}, the polynomial coefficients can be determined by using Eqs. (6*b*) and (6*c*) and Eq. (6*a*) can be thus determined. Since the right side of Eq. (6*a*) only includes the standard normal variable, the skewness α_{3S} and kurtosis α_{4S} of $S_u(u)$ can be easily obtained as

$$\alpha_{3S}=6k_1^2l_1+8l_1^3+72k_1k_2l_1+270k_2^2l_1 \tag{9a}$$

$$\alpha_{4S}=3(k_1^4+20k_1^3k_2+210k_1^2k_2^2+1260k_1k_2^3+3465k_2^4)$$
$$+12l_1^2(5k_1^2+5l_1^2+78k_1k_2+375k_2^2) \tag{9b}$$

Obviously, α_{3S} and α_{4S} should be equal to α_{3X} and α_{4X}, respectively, if $S_u(u)$ is accurate. α_{3S} and α_{4S} obtained using the present method are depicted in Fig. 3, together with the accurate ones and those obtained using the Fisher-Cornish expansion and Winterstein formula. In Fig. 3, the following two relationships of α_{3X} and α_{4X} are investigated

- Case 1, $\alpha_{4X}=2.7+1.5\alpha_{3X}^2$; and
- Case 2, $\alpha_{4X}=4+2\alpha_{3X}^2$.

One can clearly see from Fig. 3 that the relationships between α_{3S} and α_{4S} obtained by Fisher-Cornish expansion and Winterstein formula differ greatly from the accurate ones, while α_{3S} and α_{4S} obtained by the proposed formula are in close agreement with the accurate ones.

Thus, Eq. (8) is the suggested simple and accurate fourth-moment standardization function. For a random variable, if the first four moments can be obtained, the *x*-*u* and *u*-*x* transformation can be realized with Eqs. (8) and (6), respectively. Using the proposed method, the structural reliability analysis including the random variables with unknown CDFs/PDFs can be conducted without using the Rosenblatt transformation.

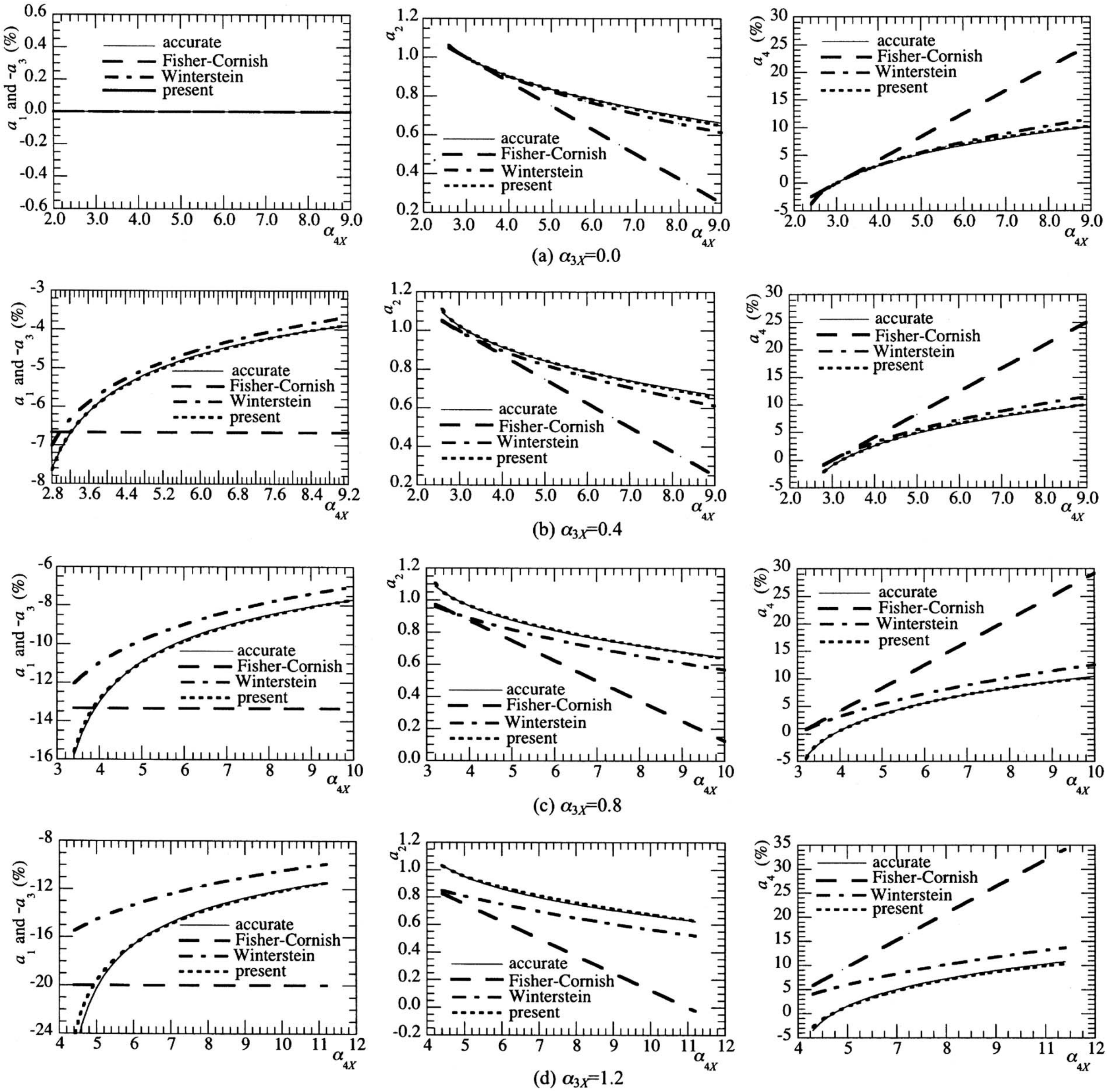


Fig. 2. Comparisons of the determination of polynomial coefficients using four different methods

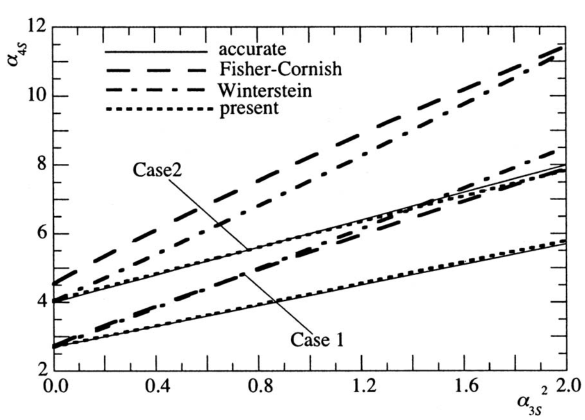
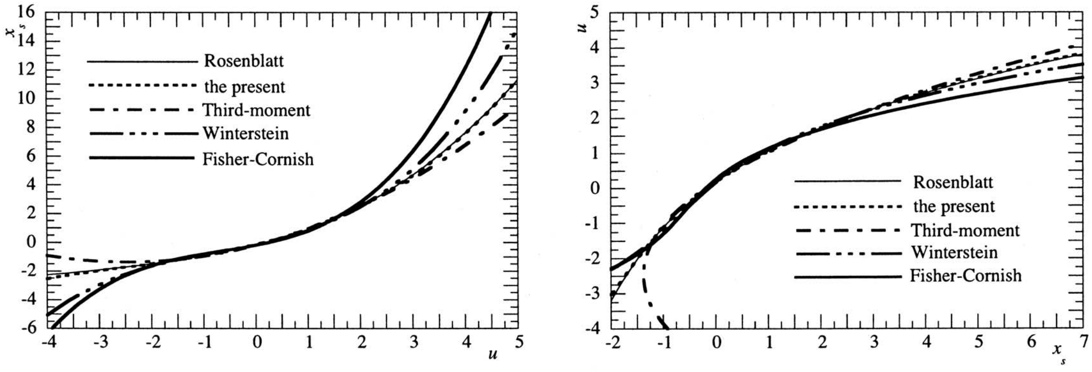


Fig. 3. Comparison of α_{4S} and α_{3S} using four different methods

Reliability Analysis Including Random Variables with Unknown CDFs/PDFs

Using the first four central moments of an arbitrary random variable x (continuous or discontinuous) with unknown CDFs/PDFs, a standard normal u can be obtained using Eq. (8), and a random variable x' corresponding to u can be obtained from Eq. (6). Since u is a continuous random variable, x' will also be a continuous random variable. Although x and x' are different random variables, they correspond to the same standard normal random variable, and have the same fourth central moment and the same statistical information source. Therefore, $f(x')$ can be considered to be an anticipated PDF of x. Using this PDF, reliability analysis including random variables with unknown CDFs/PDFs will be possible. Because the u-x and x-u transformations are realized directly by using Eqs. (6) and (8), the specific form of $f(x')$ is not required in FORM/SORM. Assuming that the random variables with unknown CDFs/PDFs are independent of those that have CDFs/PDFs and are independent of each other as well, from Eqs.



(a)

(b)

Fig. 4. *u-x* and *x-u* transformations for Gumbel random variable

(6) and (8), the element of the Jacobian matrix corresponding to a random variable x with an unknown CDF/PDF can be given as

$$J_{ii} = \frac{\partial u_i}{\partial x_i} = \frac{1}{\sigma[k_1 + 2l_1 u_i + 3k_2 u_i^2]} \tag{10}$$

For a reliability analysis with all the random variables that have a known CDF/PDF, the analysis can be conducted using the general FORM procedure (Ang and Tang 1984). For a reliability analysis including random variables with unknown CDFs/PDFs, the random variables $\mathbf{X}$ can be divided into two groups $\mathbf{X}=[\mathbf{X}_1,\mathbf{X}_2]$, where $\mathbf{X}_1$ are the random variables that have known CDFs/PDFs, and $\mathbf{X}_2$ are those with unknown CDFs/PDFs. For $\mathbf{X}_1$, the normal transformation and Jacobian matrix are conducted using the Rosenblatt transformation, and for $\mathbf{X}_2$, the normal transformation is conducted using Eq. (8), and the Jacobian matrix are obtained using Eq. (10). Then, the procedure is identical to that of the general FORM, with the exception of the conduction of the normal transformation and the computation of the elements of the Jacobian matrix corresponding to the random variables with unknown CDFs/PDFs. Therefore, the reliability analysis including random variables with unknown CDFs/PDFs using the proposed method requires only minimal extra computational effort, compared to the general FORM procedure.

When random variables that have unknown CDFs/PDFs are contained in a performance function with strong nonlinearity, for which a more accurate method such as SORM is required, the proposed method can also be applied. In such a case, the computational procedure is identical to that of general SORM with the exception of the *u-x* and *x-u* transformations and the computation of the elements of the Jacobian matrix corresponding to the random variables with unknown CDFs/PDFs.

Numerical Examples

u-x and x-u Transformations for Random Variables with Known CDFs/PDFs

In evaluating a normal transformation technique, the first concern could be how the relation between non-normal and normal variables is described by the technique. Suppose a random variable x is known to have a PDF, $f(x)$, the *u-x* and *x-u* transformations can be obtained by using the proposed fourth-moment standardization function or the other aforementioned normal transformation techniques. Since the Rosenblatt transformation completely preserves the known marginal distribution, that is, $F(x)=\Phi(u)$, it is used herein as the benchmark in performance evaluation for the other normal transformation techniques.

The first example considers a Gumbel random variable that has the following PDF:

$$f(x) = \frac{1}{\beta}\exp\left[-\exp\left(\frac{\alpha-x}{\beta}\right)+\left(\frac{\alpha-x}{\beta}\right)\right] \tag{11}$$

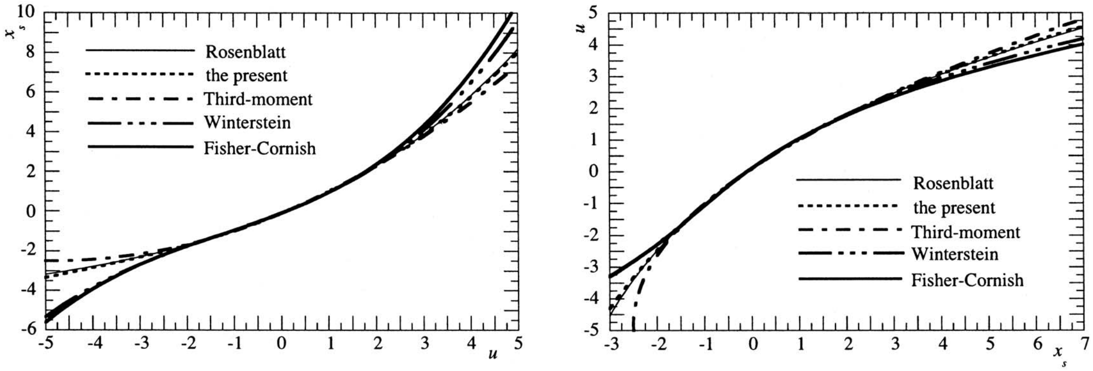
For $\alpha=0.550$ and $\beta=0.780$, the mean value, standard deviation, skewness, and kurtosis are obtained as $\mu_X=1$, $\sigma_X=1$, $\alpha_{3X}=1.140$, and $\alpha_{4X}=5.4$, respectively.

The variations of the *u-x* transformation function with respect to u and the variations of the *x-u* transformation function with respect to x_s are shown in Figs. 4(a and b), respectively, for the results obtained using the Rosenblatt transformation, the present fourth-moment transformation, the third-moment transformation (Zhao and Ono 2000), the Fisher-Cornish expansion, and the Winterstein formula. Fig. 4 reveals the following:

1. The results of the Fisher-Cornish expansion exhibit the greatest differences from the results obtained by the Rosenblatt transformation, especially when the absolute value of u or x_s is large.
2. Since only the information of the first three central moments is used in the third-moment transformation, the method yields significant errors when the absolute value of u or x_s is large for this example.
3. The transformation function obtained using the Winterstein formula provides good results when the absolute value of u or x_s is small, while when the absolute value of u or x_s is large, the results obtained from the Winterstein formula differ greatly from those obtained using the Rosenblatt transformation.
4. The proposed method performs better than the third-moment transformation, the Fisher-Cornish expansion, and the Winterstein formula, and the results of the proposed transformation coincide with those of the Rosenblatt transformation throughout the entire investigation range.

The second example considers a lognormal random variable with parameters $\lambda=2.283$ and $\zeta=0.198$. The mean value, standard deviation, skewness, and kurtosis are obtained as $\mu_X=10.0$, $\sigma_X=2.0$, $\alpha_{3X}=0.608$, and $\alpha_{4X}=3.664$, respectively.

The variations of the *u-x* and *x-u* transformation are shown in Figs. 5(a and b), respectively, for the results obtained using the



(a) (b)

Fig. 5. *u-x* and *x-u* transformations for lognormal random variable

Rosenblatt transformation, the present fourth-moment transformation, the third-moment transformation, the Fisher-Cornish expansion, and the Winterstein formula. Again, one can clearly see from Fig. 5 that the proposed method performs better than the third-moment transformation, the Fisher-Cornish expansion, and the Winterstein formula, and the results of the proposed transformation coincide with those of the Rosenblatt transformation throughout the entire investigation range.

Reliability Analysis Including Random Variables with Unknown CDF/PDFs

The third example considers the following performance function, which is an elementary reliability model that has several applications:

$$G(X) = dR - S \tag{12}$$

where R=resistance having μ_R=500 and σ_R=100; S=load with a coefficient of variation of 0.4; and d=modification of R having normal distribution, μ_d=1 and σ_d=0.1.

The following six cases are investigated under the assumption that R and S follow different probability distributions:

- Case 1: R is Gumbel (Type I-largest) and S is Weibull (Type III-smallest);,
- Case 2: R is gamma and S is normal;
- Case 3: R is lognormal and S is gamma;
- Case 4: R is lognormal and S is Gumbel;
- Case 5: R is Weibull and S is lognormal; and
- Case 6: R is Frechet (Type II-largest) and S is exponential.

Because all of the random variables in the performance function have known CDFs/PDFs, the first-order reliability index for the six cases described above can be readily obtained using FORM. In order to investigate the efficiency of the proposed reliability method, including random variables with unknown CDFs/PDFs, the CDF/PDF of random variable R in the six cases is assumed to be unknown, and only its first four moments are known. With the first four moments, the *u-x* and *x-u* transformations in FORM can be performed easily using the proposed method, and then the first-order reliability index, including random variables that have unknown CDFs/PDFs, can also be readily obtained.

The skewness and kurtosis of R corresponding to cases 1–6 are easily obtained as 1.14 and 5.4, 0.4 and 3.24, 0.608 and 3.664, −0.352 and 3.004, and 2.353 and 16.43, respectively. The first-order reliability index obtained using the CDF/PDF of R, and using only the first four moments of R, are depicted in Fig. 6 for mean values of S in the range of 100–500. Fig. 6 reveals that, for all six cases, the results of the first-order reliability index obtained using only the first four moments of R are in agreement with those obtained using the CDF/PDF of R. This is to say that the proposed method is accurate enough to include random variables with unknown CDFs/PDFs.

For Case 4, the detailed results obtained while determining the design point using the CDF/PDF of R and using the first four moments of R are listed in Table 1. Table 1 shows that the checking point (in original and standard normal space), the Jacobian, and the first-order reliability index obtained in each iteration using the first four moments of R (columns 6–9) are generally close to those obtained in each iteration using the CDF/PDF of R (columns 2–5).

As an application of Example 3, the fourth example considers the following performance function of an H-shaped steel column

$$G(X) = AY - C \tag{13}$$

where A=section area; Y=yield stress; and C is the compressive stress. The CDFs of A and Y are unknown. The only information that is known about them is their first four moments (Ono et al. 1986), i.e., μ_A=71.656 cm^2; σ_A=3.691 cm^2; α_{3A}=0.709; α_{4A}=3.692; μ_Y=3.055t/cm^2; σ_Y=0.364; α_{3Y}=0.512; and α_{4Y}=3.957. C is assumed as a lognormal variable with a mean value of μ_C=100t and a standard deviation of σ_C=40t. The skew-

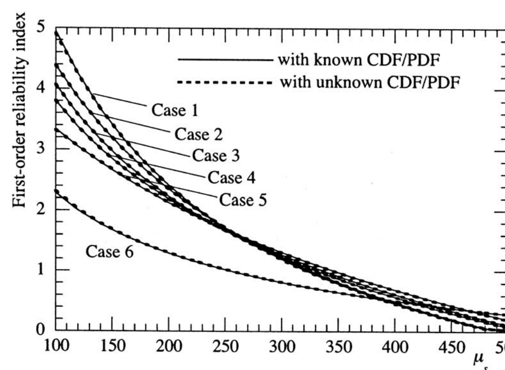


Fig. 6. Comparisons of first-order reliability index with known and unknown CDFs/PDFs

Table 1. Comparisons of FORM Procedure with Known and Unknown CDF/PDFs for Example 3

Iteration	Using CDF/PDF: Checking point (x)	(u)	Jacobian (dx/du)	β	Using the first four moments: Checking point (x)	(u)	Jacobian (dx/du)	β
1	1.000	0.000	0.100	2.254	1.000	0.000	0.100	2.254
	500.000	0.099	99.021		500.000	0.099	99.017	
	200.000	0.177	76.492		200.000	0.177	76.492	
2	0.916	−0.837	0.100	2.219	0.916	−0.836	0.100	2.224
	326.151	−2.058	64.592		326.142	−2.055	65.484	
	284.326	1.102	107.917		284.319	1.102	107.914	
5	0.947	−0.528	0.100	2.190	0.947	−0.528	0.100	2.190
	403.013	−0.990	79.814		403.132	−0.989	79.752	
	381.741	1.881	143.638		381.845	1.881	143.674	

ness and kurtosis of C can soon be obtained as $\alpha_{3C}=1.264$ and $\alpha_{4C}=5.969$.

Although the CDFs of A and Y are unknown, since the first four moments are known, the x-u and u-x transformations can be easily realized using the present method instead of the Rosenblatt transformation, and FORM can be readily conducted with results of $\beta_{\text{FORM}}=2.079$ and $P_f=0.0188$. Furthermore, using Eq. (6), the random sampling of A and Y can be easily generated without using their CDFs, and thus, the Monte Carlo simulation (MCS) can be approximately conducted. The probability of failure of this performance function is obtained as $P_f=0.0183$, and the corresponding reliability index is equal to 2.090 when the number of samples taken is 500,000. The coefficient of variation (COV) of this MCS estimate is 1.035%. One can see that the results obtained by the two methods for this example are almost the same.

Application in Point-Fitting SORM

The fifth example considers the following performance function; a plastic collapse mechanism of an elastoplastic frame structure with one story and one bay, as shown in Fig. 7

$$G(X)=M_1+3M_2+2M_3-15S_1-10S_2 \tag{14}$$

The variables of M_i and S_i are statistically independent and lognormally distributed, and have means of $\mu_{M1}=\mu_{M2}=\mu_{M3}=500$ ft K, $\mu_{S1}=50$ K, and $\mu_{S2}=100$ K, respectively, and COVs of $V_{M1}=V_{M2}=V_{M3}=0.15$, $V_{S1}=0.30$, and $V_{S2}=0.20$, respectively.

Because all of the random variables in the performance function shown in Eq. (14) have a known CDF/PDF, the reliability index can be readily obtained using FORM/SORM. The FORM reliability index is $\beta_{\text{FORM}}=2.851$, which corresponds to a failure probability of $P_f=2.181\times10^{-2}$. Using the MCS method, the reliability index is obtained as $\beta_{\text{MCS}}=2.794$, and the corresponding probability of failure is equal to 2.603×10^{-3}. The COV of this MCS estimate is 0.875%. Using the point-fitting SORM (Zhao and Ono 1999), the point-fitted performance function is obtained as

$$G(u)=1219.14+73.66u_1+218.97u_2+146.75u_3-151.26u_4 -184.59u_5+5.20u_1^2+14.17u_2^2+9.89u_3^2-57.05u_4^2 -25.61u_5^2 \tag{15}$$

The average curvature radius is obtained as $R=56.62$ and the second-order reliability index (Zhao et al. 2002) is obtained as $\beta_{\text{SORM}}=2.816$, which corresponds to a failure probability of $P_f=2.44\times10^{-3}$.

In order to investigate the application of the proposed reliability method, including random variables with unknown CDF/PDFs to the point-fitting SORM, the CDFs/PDFs of random variables S_1 and S_2 are assumed to be unknown, and only the first four moments are known. Using the first four moments, the u-x and x-u transformations can be performed easily using the proposed method, and then the point-fitting SORM, including random variables with unknown CDFs/PDFs, can also be performed easily. The point-fitted performance function is obtained as

$$G'(u)=1233.15+73.65u_1+218.97u_2+146.75u_3-166.77u_4 -188.82u_5+5.20u_1^2+14.17u_2^2+9.89u_3^2-52.92u_4^2 -25.26u_5^2 \tag{16}$$

The average curvature radius is given as $R=-63.31$ and the second-reliability indices is obtained as $\beta_{\text{SORM}}=2.817$, which corresponds to a failure probability of $P_f=2.43\times10^{-3}$. One can see that the results obtained using the first four moments of S_1 and S_2 are very close to those obtained using the CDFs/PDFs of S_1 and S_2. This is to say that the proposed u-x and x-u transformations are applicable to the point-fitting SORM.

The sixth example considers the following strong nonlinear performance function

$$G(x)=x_1^4+x_2^2-50 \tag{17}$$

where x_1 and x_2=statistically independent; x_1=lognormal variable with mean value of 5 and COV of 0.2; and x_2=Gumbel variable with mean value of 10 and COV of 1.

Because x_1 and x_2 have a known CDF/PDF, using FORM, SORM, and MCS, the reliability indices can be readily obtained as: $\beta_{\text{FORM}}=3.254$; $\beta_{\text{SORM}}=3.562$; and $\beta_{\text{MCS}}=3.570$ (the COV of

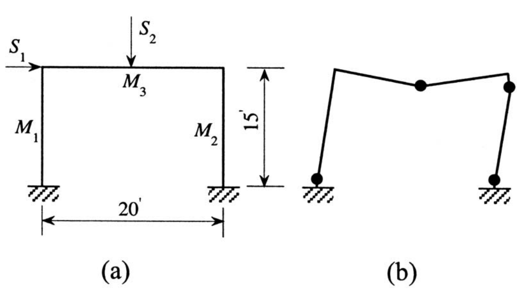


Fig. 7. One-story one-bay frame

MCS is 3.35%). In order to investigate the application of the proposed reliability method, including random variables with unknown CDFs/PDFs to FORM, SORM, and MCS, the CDFs/PDFs of random variables x_1 and x_2 are assumed to be unknown, and only the first four moments are known. The results are obtained as: $\beta_{FORM}=3.220$; $\beta_{SORM}=3.513$; and $\beta_{MCS}=3.509$ (the COV of MCS is 2.98%). Apparently, the results obtained by using the first four moments of x_1 and x_2 are close to those obtained using the CDFs/PDFs of x_1 and x_2.

Conclusions

1. A simple explicit fourth-moment standardization function is proposed. It is found to be accurate enough to include independent random variables with unknown CDFs/PDFs in reliability analysis using FORM/SORM.
2. Since the proposed fourth-moment formula can give a good approximation for the polynomial coefficients using the first four central moments, the present method provides more appropriate *u-x* and *x-u* transformation results compared to the third-moment function, Fisher-Cornish expansion, or Winterstein formula.

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Notation

The following symbols are used in this paper:

a, b, c, and d = coefficients in the PDF of Pearson system;
a_1, a_2, a_3, a_4 = polynomial coefficients used in the third-order polynomial normal transformation;
$f(\mathbf{X})$ = joint probability density function of $\mathbf{X}$;
$G(\mathbf{X})$ = performance function;
h_3, h_4 = Hermite series coefficients [Eq. (4)];
$\tilde{h}_3, \tilde{h}_4, \tilde{k}$ = Hermite series coefficient [Eq. (5)];
l_1, l_2, k_1, k_2 = coefficients of Eq. (6);
P_f = probability of failure;
R = resistance;
S = load;
$S_u(u)$ = the third polynomial of u;
U = standard normal random variables;
u = standard normal random variable;
V = coefficient of variation;
$\mathbf{X}$ = random variables;
x_s = random variable corresponding to x with its mean value=0 and standard deviation =1;
α_{3X} = coefficient of skewness of random variable x;
α_{4X} = coefficient of kurtosis of random variable x;
β_{FORM} = first-order reliability index;
β_{SORM} = second-order reliability index;
μ = mean value;
σ = standard deviations; and
$\Phi(x)$ = standard normal probability distribution with argument x;

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